LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **STATISTICS**

FIRST SEMESTER – NOVEMBER 2007

ST 1809 - MEASURE AND PROBABILITY

BB 11

Date : 27/10/2007 Dept. No. Max. : 100 Marks

Time : 1:00 - 4:00

SECTION A

Answer ALL questions **2 * 10 = 20**

1. Let $\{A_n, n \geq 1\}$ be a sequence of numbers defined as follows.

$$A_n = \begin{cases} \left(\frac{-1}{n}, 1\right] & if\ n \quad is \quad odd \\ \left(-1, \frac{1}{n}\right] & if \quad n \quad is \quad even \end{cases}$$

Find $\lim_{n \to \infty} \sup A_n$ and $\lim_{n \to \infty} \inf A_n$

2. Show that every σ field is a field.
3. Define Counting measure.
4. Give the definition of a simple Borel measurable function.
5. State Radon – Nikodym theorem.
6. Give an example to show that two different random variables can have same distribution function.
7. If the random variable X takes only positive integer values then show that

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n)$$

8. If $Y_1 \leq Y_2$ then show that E (Y_1 | $\dot{g}$) ≤ E (Y_2 | $\dot{g}$).
9. Define Convergence in distribution of a sequence of random variables.
10. State the Central Limit Theorem of a sequence of random variables.

SECTION B

Answer any FIVE questions **5 * 8 = 40**

11. Let $\{X_n, n \geq 1\}$ be a sequence of real numbers and let A_n = (-∞, X_n). What is the connection between $\lim_{n \to \infty} \sup A_n$ and $\lim_{n \to \infty} \sup X_n$
12. Let $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ be two increasing sequences of sets defined on $(\Omega, \Im, P)$ such that $\lim_{n \to \infty} A_n \subset \lim_{n \to \infty} B_n$ then show that $\lim_{n \to \infty} P(A_n) \leq \lim_{n \to \infty} P(B_n)$
13. State and prove Fatou's lemma.
14. State and prove Minkowski's inequality.

15. a.) Define discrete and continuous random variable.

b.) Let X be a random variable with the distribution function

$$F(x) = \begin{cases} 0 & , \quad x \prec 0 \\ \frac{1}{4} & , \quad 0 \le x \prec 1 \\ \frac{1}{2} & , \quad 1 \le x \prec 2 \\ \frac{1}{2} + \frac{x-2}{2} & , \quad 2 \le x \prec 3 \\ 1 & , \quad x \ge 3 \end{cases}$$

Show that X is neither discrete nor continuous.

16. Let X be a random object defined on $(\Omega, \Im, P)$ and let $B \in \Im$. Show that E (I_B | X = x) = P (B | X =x) a.e. [P].

17. Let $Y_n \ge 0 \; \forall \; n \ge 1$ be a sequence of random variables such that $Y_n \uparrow Y$. Show that E (Y_n | $\dot{g}$) ↑ E (Y | $\dot{g}$) a.e. [P].

18. Let $\{Y_n, n \ge 1\}$ be a sequence of iid random variables with the probability density function $f(y;\theta) = e^{-(y-\theta)}, y \ge \theta$. Let X_n = min ($X_1, X_2, \ldots, X_n$). Show that $X_n \xrightarrow{a.s.} \theta$.

SECTION C

Answer any TWO questions **2 * 20 = 40**

19. State and prove Basic Integration Theorem.

20. a.) State and prove Additivity theorem of integral of Borel measurable functions.

b.) Let μ be a measure and λ_1 and λ_2 be signed measures defined on the σ field $\Im$ of subsets of Ω. Prove that if $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$ (12+8)

21. a.) A coin is tossed independently and indefinitely. Define the event A_{2n} as in the $2n^{th}$ toss equalization of head and tail occurs. Show that P (A_{2n} occurring infinitely often) is zero or one corresponding to the coin being biased or unbiased respectively.

b.) State and prove Weak law of large numbers. (14+6)

22. a.) Let $(\Omega, \Im, P)$ be a probability space and let $\dot{g}$ be sub σ field of $\Im$. Fix $B \in \Im$. Show that there is a function P(B | $\dot{g}$) : (Ω, $\dot{g}$) → (R,B(R)) called the probability of B | $\dot{g}$ such that $P(C \cap B) = \int_c P(B \mid g) \, dP \; \forall \; c \in g$. Further show that any two such functions must coincide and P(B | $\dot{g}$) = E (I_B | $\dot{g}$) a.e. [P].

b.) Show that $X_n \xrightarrow{a.s} X$ does not imply $X_n \xrightarrow{q.m.} X$. Also show that $X_n \xrightarrow{q.m.} X$ does not imply $X_n \xrightarrow{a.s} X$. (12+8)

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